INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2012-13
Statistics - III, Midterm Examination, September 12, 2012

1. Suppose $Z_i \sim N(0, \sigma^2)$, $1 \le i \le 3$ are independent normally distributed random variables and let $0 < \rho < 1$. Define $X_1 = Z_1$, $X_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2$ and $X_3 = \rho X_2 + \sqrt{1-\rho^2} Z_3$. Let $\mathbf{X} = (X_1, X_2, X_3)'$ and Σ denote the covariance matrix of $\mathbf{X}$.
(a) Find the probability distribution of $\mathbf{X}$.
(b) Find the probability distribution of $\mathbf{X}'\Sigma^{-1}\mathbf{X}$.
(c) Find the probability distribution of $\mathbf{X}'C'PC\mathbf{X}$ where $C'C = \Sigma^{-1}$ and P is a symmetric idempotent matrix of rank 2.
(d) Find the conditional distribution of $\mathbf{X}'\Sigma^{-1}\mathbf{X}$ given $\sum_{i=1}^{3} Z_i = 0$. [15]

2. Let A^- be a generalized inverse of a matrix A. Show that
(a) $\text{Rank}(A) = \text{Rank}(AA^-) = \text{Rank}(A^-A) \le \text{Rank}(A^-)$;
(b) $A(A'A)^-A'$ is unique even though $(A'A)^-$ may not be unique;
(c) $\mathcal{M}_C(A(A'A)^-A') = \mathcal{M}_C(A)$. [10]

3. Consider the model $\mathbf{Y} = X\beta + \epsilon$, where $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$ and $X_{n\times p}$ may not have full column rank. Let $\hat{\beta}$ is any least squares estimator of β. Derive the joint distribution of $\mathbf{a}'\hat{\beta}$ and the Residual Sum of Squares when $\mathbf{a}'\beta$ is estimable. [10]

4. Consider the following model:
$y_1 = \alpha + \delta + \epsilon_1$
$y_2 = \delta + \gamma + \epsilon_2$
$y_3 = -\alpha - 2\delta - \gamma + \epsilon_3$
$y_4 = -\alpha - \delta + \epsilon_4$,
where α, δ, γ are unknown constants and ϵ_i are uncorrelated random variables having mean 0 and variance σ^2.
(a) Show that $\alpha - \gamma$ is estimable. What is its BLUE?
(b) Does there exist a BLUE for $\alpha + \gamma$? Justify.
(c) Find an unbiased estimate of σ^2. [15]